

# Homology of saddle point reduction and applications to resonant elliptic systems <sup>\*</sup>

Chong Li <sup>a</sup>      Shibo Liu <sup>b†</sup>

<sup>a</sup> *Institute of Mathematics, Chinese Academy of Sciences, Beijing 100190, P.R. China*

<sup>b</sup> *School of Mathematical Sciences, Xiamen University, Xiamen 361005, PR China*

In the setting of saddle point reduction, we prove that the critical groups of the original functional and the reduced functional are isomorphic. As application, we obtain two nontrivial solutions for elliptic gradient systems which may be resonant both at the origin and at infinity. The difficulty that the variational functional does not satisfy the Palais-Smale condition is overcome by taking advantage of saddle point reduction. Our abstract results on critical groups are crucial.

*Keywords:* Critical groups; saddle point reduction; Künneth formula; resonant elliptic systems

*MSC2000:* 58E05; 35J60

## 1 Introduction

Infinite dimensional Morse theory (see [9, 27] for a systematic exploration) is very useful in obtaining multiple solutions for nonlinear variational problems. The central concept in this theory is the critical group  $C_*(f, u)$  for a  $C^1$ -functional  $f : X \rightarrow \mathbb{R}$  at an isolated critical point  $u$ . The critical group describes the local property of  $f$  near the critical point  $u$ . On the other hand, Bartsch and Li [4] introduced the critical group  $C_*(f, \infty)$  of  $f$  at infinity, which describes the global property of the functional  $f$ .

With these concepts we have the Morse inequalities

$$\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1+t)Q(t), \quad (1.1)$$

where  $Q$  is a formal series with nonnegative integer coefficients,

$$M_q = \sum_{f'(u)=0} \text{rank } C_q(f, u), \quad \beta_q = \text{rank } C_q(f, \infty).$$

In most applications, we may distinguish critical points using critical group, and we may find new critical points using the Morse inequality. Therefore, the study of the critical group is very important.

---

<sup>\*</sup>Supported by NSFC (11071237, 11171204) and RFDP (20094402110001).

<sup>†</sup>email: liusb@xmu.edu.cn

In 1979, Amann [1] established the theory of saddle point reduction (also called Lyapunov-Schmidt reduction in some literature). Since then, saddle point reduction becomes an important method in critical point theory, and has been widely applied to various nonlinear boundary value problems [7, 10, 12, 18, 23, 25, 26].

Let  $(X, \langle \cdot, \cdot \rangle)$  be a separable Hilbert space with norm  $\|\cdot\|$ , and  $f \in C^1(X, \mathbb{R})$ . The basic assumption in saddle point reduction is the following

$(A_{\pm})$   $X^{\pm}$  are closed subspaces of  $X$  such that  $X = X^{-} \oplus X^{+}$ , and there exists a real number  $\kappa > 0$  such that

$$\pm \langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2$$

for all  $v \in X^{-}$  and  $w_1, w_2 \in X^{+}$ .

Then by saddle point reduction, there exist  $\psi \in C(X^{-}, X^{+})$  and  $\varphi \in C^1(X^{-}, \mathbb{R})$  such that  $\bar{v}$  is a critical point of  $\varphi$  if and only if  $\bar{v} + \psi(\bar{v})$  is a critical point of  $f$ ; moreover we have

$$\varphi(v) := f(v + \psi(v)) = \max_{w \in X^{+}} f(v + w), \quad \forall v \in X^{-} \quad (1.2)$$

for case  $(A_{-})$  and with ‘max’ replaced by ‘min’ for case  $(A_{+})$ , see [8] for a good proof of these results. Thus, to find critical points of  $f$  we may consider the reduced functional  $\varphi$ . Since  $\varphi$  is defined on a subspace, it should be easier to study.

As mentioned before, Morse theory is a powerful tool in the study of variational problems. If we want to apply Morse theory, naturally we need to study the relation between the critical group of  $\varphi$  and that of  $f$ . In our previous work [22, 24], we proved the results described in the following theorem.

**Theorem A.** *Let  $X$  be a separable Hilbert space and  $f \in C^1(X, \mathbb{R})$ .*

- (i) *If  $(A_{+})$  holds,  $f$  satisfies the Palais-Smale (PS) condition with critical values bounded from below, then  $C_q(f, \infty) \cong C_q(\varphi, \infty)$  for  $q = 0, 1, 2, \dots$ .*
- (ii) *If  $(A_{-})$  holds and  $\mu = \dim X^{+} < \infty$ ,  $f$  satisfies (PS) condition with critical values bounded from below, then*

$$C_q(f, \infty) \cong C_{q-\mu}(\varphi, \infty), \quad \text{for } q = 0, 1, 2, \dots \quad (1.3)$$

- (iii) *If  $(A_{+})$  holds and  $\bar{v} \in X^{-}$  is an isolated critical point of  $\varphi$ , then*

$$C_q(f, \bar{v} + \psi(\bar{v})) \cong C_q(\varphi, \bar{v}), \quad \text{for } q = 0, 1, 2, \dots$$

In view of Theorem A, we naturally expect that in case  $(A_{-})$  there should have a relation similar to (1.3) for the critical groups at isolated critical points. In this paper, we will establish such a relation in the following theorem.

**Theorem 1.1.** *Let  $X$  be a separable Hilbert space and  $f \in C^1(X, \mathbb{R})$ . Assume  $(A_{-})$  holds and  $\mu = \dim X^{+} < \infty$ ;  $\bar{v} \in X^{-}$  is an isolated critical point of  $\varphi$  such that  $\varphi(\bar{v})$  is an isolated critical value. If moreover  $\varphi$  satisfies the (PS) condition, then*

$$C_q(f, \bar{v} + \psi(\bar{v})) \cong C_{q-\mu}(\varphi, \bar{v}), \quad \text{for } q = 0, 1, 2, \dots$$

This theorem and Theorem A completely describe the relation between the critical groups of the original functional  $f$  and the reduced functional  $\varphi$ .

Under the assumption of Theorem 1.1,  $\bar{v} + \psi(\bar{v})$  is an isolated critical point of  $f$ . It is also well known that if  $f$  satisfies the  $(PS)$  condition, so does  $\varphi$ , see [3, Lemma 1]. However, we emphasize that in Theorem 1.1 it is not necessary to require that  $f$  satisfies  $(PS)$ . This is very important in applications.

Let  $\text{ind}(T, u)$  denotes the Leray–Schauder index for  $T : X \rightarrow X$  (a compact perturbation of the identity map) at its isolated zero point  $u$  and assume that  $\nabla f$  is a compact perturbation of the identity. Similar to [22, Corollary 2.4], as a corollary of Theorem 1.1 and the Poincaré–Hopf formula for  $C^1$ -functional [14, Theorem 3.2]

$$\text{ind}(\nabla f, u) = \sum_{q=0}^{\infty} (-1)^q \text{rank } C_q(f, u),$$

if  $(A_-)$  holds and  $\mu = \dim X^+ < \infty$ , we have

$$\text{ind}(\nabla f, \bar{v} + \psi(\bar{v})) = (-1)^\mu \text{ind}(\nabla \varphi, \bar{v}). \quad (1.4)$$

The corresponding result for case  $(A_+)$ , namely [22, Corollary 2.4], is originally due to Lazer and McKenna [13]. As far as we know, the identity (1.4) does not appear elsewhere.

Our investigation of the relation between the critical groups of the original functional and the reduced functional is motivated by the study of multiple solutions for nonlinear boundary value problems. In the second part of this paper, as applications of our abstract results we consider asymptotically linear elliptic systems. Such problems have attracted some attentions in recent years, see [10, 12, 11, 31, 32].

Our assumptions on the nonlinearity are so weak that the corresponding Euler–Lagrange functional does not satisfy the  $(PS)$  condition. Nevertheless, using some idea from [23, 25], by taking advantage of saddle point reduction we can overcome this difficulty. As we will see in Remark 4.10, because the asymptotic limits may be different variable matrices, the local linking argument used in [23, 25] does not apply. To prove our multiplicity results (Theorems 3.3 and 3.4), Theorem A (iii) and Theorem 1.1 are crucial.

## 2 Proof of Theorem 1.1

Let  $X$  be a Banach space and  $f \in C^1(X, \mathbb{R})$ . Let  $u$  be an isolated critical point of  $f$  with critical value  $c = f(u)$ ,  $\Omega$  be an arbitrary neighborhood of  $u$ . Then the group

$$C_q(f, u) = H_q(f_c \cap \Omega, (f_c \cap \Omega) \setminus \{u\}), \quad q = 0, 1, 2, \dots$$

is called the  $q^{\text{th}}$  critical group of  $f$  at  $u$ . Here  $f_c = f^{-1}(-\infty, c]$ ,  $H_q(A, B)$  stands for the  $q^{\text{th}}$  singular relative homology group of the topological pair  $(A, B)$  with coefficients in a field  $\mathcal{G}$ . By the excision property of homology, the critical groups of  $f$  at  $u$  described the local property of  $f$  near  $u$ .

If  $f$  satisfies the  $(PS)$  condition and the critical values of  $f$  are bounded from below by  $\alpha \in \mathbb{R}$ , then according to [4, Definition 3.4], the group

$$C_q(f, \infty) = H_q(X, f_\alpha), \quad q = 0, 1, 2, \dots \quad (2.1)$$

is called the  $q^{\text{th}}$  critical group of  $f$  at infinity. Since  $f$  satisfies (PS), by the deformation lemma, the right hand side of (2.1) does not depend on the choice of  $\alpha$ . Since all critical points of  $f$  are contained in  $X \setminus f_\alpha$ , we can say that the critical groups of  $f$  at infinity describe the global property of  $f$ .

From the definitions of critical groups, we see that analytically,  $C_*(f, u)$  is simpler than  $C_*(f, \infty)$ , because the former does not require the (PS) condition; while topologically,  $C_*(f, u)$  is more complicated than  $C_*(f, \infty)$ , because the topological pair on the right hand side of (2.1) is simpler. This explains why the results in Theorem A (i) and (ii) were proved first.

*Proof of Theorem 1.1.* Assume  $\varphi(\bar{v}) = f(\bar{v} + \psi(\bar{v})) = a$ . Since  $a$  is an isolated critical value of  $\varphi$ , there is an  $\varepsilon > 0$  such that  $\varphi$  has no critical value in  $(a, a + \varepsilon]$ . Since  $\varphi$  satisfies (PS), by the second deformation lemma [9, 30], there is a continuous  $\eta : [0, 1] \times \varphi_{a+\varepsilon} \rightarrow \varphi_{a+\varepsilon}$  such that

$$\left. \begin{aligned} \eta(0, u) &= u, & u &\in \varphi_{a+\varepsilon}, \\ \eta(1, \varphi_{a+\varepsilon}) &\subset \varphi_a, \\ \eta(t, u) &= u, & (t, u) &\in [0, 1] \times \varphi_a. \end{aligned} \right\} \quad (2.2)$$

Let  $O \subset \varphi_{a+\varepsilon}$  be a neighborhood of  $\bar{v}$  such that  $\varphi$  has no critical point in  $O \setminus \{\bar{v}\}$ , and set

$$U = \left( \bigcup_{t \in [0, 1]} \eta(t, O) \right) \cup \varphi_a.$$

Then  $U$  is an  $\eta$ -invariant neighborhood of  $\bar{v}$ , and  $\Omega = U \times X^+$  is a neighborhood of  $(\bar{v}, \psi(\bar{v}))$ .

By the property of  $\varphi$  and  $\psi$  described in (1.2), if  $\varphi(v) \leq a$ , then for any  $w \in X^+$  we have  $f(v + w) \leq a$ . Thus, setting

$$\Theta = \{ (v, w) \mid f(v + w) \leq a, \varphi(v) > a \},$$

we have  $f_a = (\varphi_a \times X^+) \cup \Theta$ .

Under the assumption  $(A_-)$ , it has been shown in the proof of [24, Theorem 1.2] that  $f_a$  is homotopically equivalent to

$$A = (\varphi_a \times X^+) \cup \{ (v, w) \mid \varphi(v) > a, w \neq \psi(v) \}$$

via a homotopy  $F : [0, 1] \times A \rightarrow A$  constructed in that proof (using condition  $(A_-)$  and the implicit function theorem). Moreover, denoting

$$S = \{ w \in X^+ \mid w \neq 0 \},$$

a homeomorphism  $G$  between  $A$  and

$$B = (\varphi_a \times X^+) \cup ((X^- \setminus \varphi_a) \times S)$$

has also been given there. The deformations  $F$  and  $G$  have been illustrated in Figure 1, where the thick segments with endpoint  $\bar{v}$  represent the level set  $\varphi_a$ ; while the shadowed regions in the three subfigures represent the sets  $f_a$ ,  $A$  and  $B$  respectively.

Let  $\tilde{F}$  be the homotopy inverse of  $F$ , then  $\Gamma = G \circ \tilde{F}$  is a homotopic equivalence between  $f_a$  and  $B$ . By the definitions of  $F$  and  $G$  (see the proof of [24, Theorem 1.2]), we see that  $\Gamma$  does not change the  $v$ -variable. Therefore, restricting  $\Gamma$  to  $f_a \cap \Omega$ , we obtain

$$f_a \cap \Omega \approx B \cap \Omega = (\varphi_a \times X^+) \cup ((X^- \setminus \varphi_a) \cap U) \times S$$

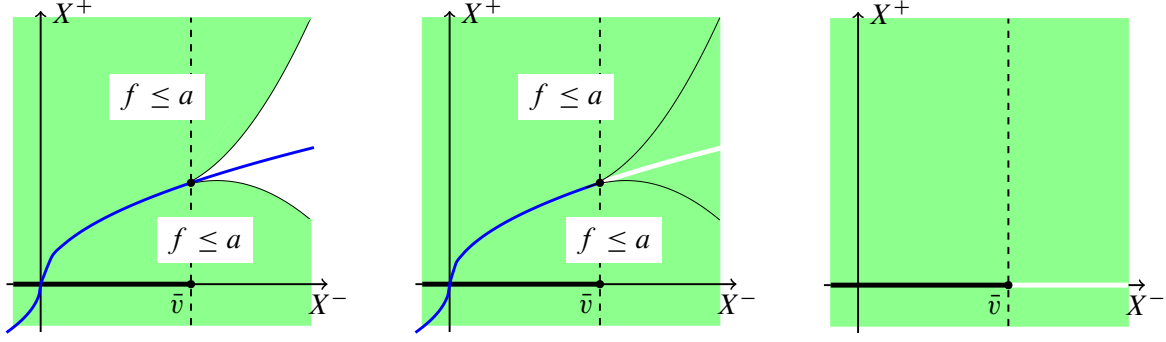


Figure 1: Deformations of the level set  $f_a$ .

Since  $U$  is invariant under the flow  $\eta$ , we can define  $H : [0, 1] \times (B \cap \Omega) \rightarrow B \cap \Omega$ ,

$$H(t, (v, w)) = \begin{cases} (v, w), & \text{if } (v, w) \in \varphi_a \times X^+, \\ (\eta(t, v), w), & \text{if } (v, w) \in ((X^- \setminus \varphi_a) \cap U) \times S. \end{cases}$$

Using (2.2), it is easy to see that  $H$  is continuous, and  $H(1, \cdot)$  is a homotopic equivalence between  $B \cap \Omega$  and  $\varphi_a \times X^+$ . Combining the above homotopies, we can deform  $f_a \cap \Omega$  to  $\varphi_a \times X^+$  continuously. The deformation maps  $(\bar{v}, \psi(\bar{v}))$  to  $(\bar{v}, 0)$ , therefore it is also a homotopic equivalence between  $(f_a \cap \Omega) \setminus (\bar{v}, \psi(\bar{v}))$  and  $(\varphi_a \times X^+) \setminus (\bar{v}, 0)$ .

Noting that  $(\varphi_a \times X^+) \setminus (\bar{v}, 0) = (\varphi_a \times S) \cup ((\varphi_a \setminus \bar{v}) \times X^+)$ , we have

$$\begin{aligned} (f_a \cap \Omega, (f_a \cap \Omega) \setminus (\bar{v}, \psi(\bar{v}))) &\simeq (\varphi_a \times X^+, (\varphi_a \times X^+) \setminus (\bar{v}, 0)) \\ &= (\varphi_a \times X^+, (\varphi_a \times S) \cup ((\varphi_a \setminus \bar{v}) \times X^+)) \\ &= (\varphi_a, \varphi_a \setminus \bar{v}) \times (X^+, S). \end{aligned}$$

Passing to homology and applying the Künneth formula, we deduce

$$\begin{aligned} C_*(f, \bar{v} + \psi(\bar{v})) &= H_*(f_a \cap \Omega, (f_a \cap \Omega) \setminus (\bar{v}, \psi(\bar{v}))) \\ &\cong H_*((\varphi_a, \varphi_a \setminus \bar{v}) \times (X^+, S)) \\ &= H_*(\varphi_a, \varphi_a \setminus \bar{v}) \otimes H_*(X^+, S) \\ &= H_{*-\mu}(\varphi_a, \varphi_a \setminus \bar{v}) = C_{*-\mu}(\varphi, \bar{v}), \end{aligned}$$

where we have used the fact that  $H_q(X^+, S) = \delta_{q,\mu} \mathcal{G}$ , since  $\dim X^+ = \mu$ .  $\square$

In critical point theory, it will be very convenient if the gradient of the functional under consideration is a compact perturbation of the identity operator. Hence, if the original functional  $f$  has this property, we hope that the reduced functional  $\varphi$  also has such property. This is true if  $\nabla f : X \rightarrow X$  maps bounded sets to bounded sets.

**Proposition 2.1** ([23, Corollary 2.2]). *Let  $X$  be a separable Hilbert space and  $f \in C^1(X, \mathbb{R})$ . Assume  $(A_+)$  or  $(A_-)$  holds. If  $\nabla f : X \rightarrow X$  is bounded and there is a compact operator  $K : X \rightarrow X$  such that  $\nabla f = \mathbf{1}_X - K$ , then there is a compact operator  $Q : X^- \rightarrow X^-$  such that  $\nabla \varphi = \mathbf{1}_{(X^-)} - Q$ .*

Finally, for the convenience of our later application, we recall a homological version of the famous three critical points theorem.

**Proposition 2.2** ([21, Theorem 2.1]). *Let  $X$  be a Banach space and  $f \in C^1(X, \mathbb{R})$  satisfy the Palais-Smale (PS) condition. Assume that  $f$  is bounded from below. If  $C_\ell(f, \mathbf{0}) \neq 0$  for some  $\ell \neq 0$ , then  $f$  has at least three critical points.*

We note that according to [9, Page 33],  $C_\ell(f, \mathbf{0}) \neq 0$  for some  $\ell \neq 0$  implies that  $\mathbf{0}$  is not a local minimizer of  $f$ , as required in the original statement of [21, Theorem 2.1].

### 3 Multiple solutions of elliptic systems

In this section, as application of our abstract results on critical groups, we consider elliptic gradient systems of the form

$$\begin{cases} -\Delta u = F_u(x, u, v), & \text{in } \Omega, \\ -\Delta v = F_v(x, u, v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded smooth domain,  $F \in C^1(\Omega \times \mathbb{R}^2, \mathbb{R})$  satisfies the linear growth condition

$$|\nabla F(x, \mathbf{z})| \leq \Lambda |\mathbf{z}|, \quad (x, \mathbf{z}) \in \Omega \times \mathbb{R}^2 \quad (3.2)$$

for some constant  $\Lambda > 0$ . Here, to simplify the notations we denote  $\mathbf{z} = (u, v)$ . The gradient is taken with respect to  $\mathbf{z}$ . Without loss of generality we may assume  $F(x, \mathbf{0}) = 0$ . We also assume  $\nabla F(x, \mathbf{0}) = \mathbf{0}$ , so that  $\mathbf{z} = \mathbf{0}$  is a trivial solution of (3.1). Therefore, we will focus on nontrivial solutions.

To apply variational methods, let  $X$  be the Hilbert space  $H_0^1(\Omega) \times H_0^1(\Omega)$  endowed with the inner product

$$\langle \mathbf{z}, \mathbf{w} \rangle = \int_{\Omega} \nabla \mathbf{z} \cdot \nabla \mathbf{w} \, dx$$

and corresponding norm  $\|\cdot\|$ , here  $\mathbf{z} = (u, v)$ ,  $\nabla \mathbf{z} = (\nabla u, \nabla v)$ , the dot  $\cdot$  represents the standard inner product in  $\mathbb{R}^2$ . Under the growth condition (3.2), the functional  $\Phi : X \rightarrow \mathbb{R}$ ,

$$\Phi(\mathbf{z}) = \frac{1}{2} \int_{\Omega} |\nabla \mathbf{z}|^2 \, dx - \int_{\Omega} F(x, \mathbf{z}) \, dx \quad (3.3)$$

is well defined and of class  $C^1$ . The critical points of  $\Phi$  are solutions of the system (3.1).

Before state our assumptions on  $F$  and our main results, let us denote by  $\mathcal{M}_2(\Omega)$  the set of those positive definite symmetric matrix functions  $A : \bar{\Omega} \rightarrow M_{2 \times 2}(\mathbb{R})$  whose entries are continuous real functions on  $\bar{\Omega}$ .

For given  $A \in \mathcal{M}_2(\Omega)$ , there is an associated weighted eigenvalue problem

$$\begin{cases} -\Delta \mathbf{z} = \lambda A(x) \mathbf{z} & \text{in } \Omega, \\ \mathbf{z} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \quad (3.4)$$

Since  $A(x)$  is positive definite, using the spectral theory of compact self-adjoint operator, it is well known that there is a complete list of distinct eigenvalues

$$0 < \lambda_1(A) < \lambda_2(A) < \dots$$

such that  $\lambda_n(A) \rightarrow +\infty$  as  $n \rightarrow \infty$ . Our multiplicity results depend on the interaction between the nonlinearity  $F$  and the eigenvalues of (3.4).

We also need the following concepts introduced by da Silva [10, Definition 1.4].

**Definition 3.1.** Let  $A, B \in \mathcal{M}_2(\Omega)$ .

- (i) We define  $A \leq B$  if  $A(x)\mathbf{z} \cdot \mathbf{z} \leq B(x)\mathbf{z} \cdot \mathbf{z}$  for all  $(x, \mathbf{z}) \in \Omega \times \mathbb{R}^2$ .
- (ii) We define  $A \preceq B$  if  $A \leq B$  and  $B - A$  is positive definite on  $\tilde{\Omega} \subset \Omega$  with  $|\tilde{\Omega}| > 0$ , where  $|\cdot|$  stands for the Lebesgue measure.

To state our assumptions on the nonlinearity  $F(x, \mathbf{z})$ , we assume that there exists  $A_0 \in \mathcal{M}_2(\Omega)$  with  $\lambda_m(A_0) = 1$  for some  $m \in \mathbb{N}$ , such that

$$G(x, \mathbf{z}) := F(x, \mathbf{z}) - \frac{1}{2}A_0(x)\mathbf{z} \cdot \mathbf{z} = o(|\mathbf{z}|^2), \quad \text{as } |\mathbf{z}| \rightarrow 0 \quad (3.5)$$

uniformly for  $x \in \Omega$ . We then assume the following conditions on  $F$ .

- $(F_0^\pm)$  There exists some  $\delta > 0$  such that  $\pm G(x, \mathbf{z}) > 0$  for  $0 < |\mathbf{z}| \leq \delta$ .
- $(F_\infty^\pm)$  There exists  $A_\infty \in \mathcal{M}_2(\Omega)$  with  $\lambda_k(A_\infty) = 1$  for some  $k \in \mathbb{N}$ , such that

$$\lim_{|\mathbf{z}| \rightarrow \infty} \left( F(x, \mathbf{z}) - \frac{1}{2}A_\infty(x)\mathbf{z} \cdot \mathbf{z} \right) = \pm\infty.$$

*Remark 3.2.* (i) If  $F \in C^2(\Omega \times \mathbb{R}^2, \mathbb{R})$  verifies  $F(x, \mathbf{0}) = 0$ ,  $\nabla F(x, \mathbf{0}) = \mathbf{0}$ , then by the Taylor formular we see that (3.5) holds with  $A_0(x)$  being the Hessian of  $F(x, \cdot)$  at  $\mathbf{z} = \mathbf{0}$ .

- (ii) If  $1 \in (\lambda_m(A_0), \lambda_{m+1}(A_0))$  for some  $m \in \mathbb{N}$ , namely the problem (3.1) is non resonant at the origin, then the condition  $(F_0^\pm)$  can be removed. The same remark applies to  $(F_\infty^\pm)$  if (3.8) holds.

For the sake of simplicity, we denote  $\lambda_n(A_0)$  by  $\lambda_n^0$ , and  $\lambda_n(A_\infty)$  by  $\lambda_n^\infty$ . Set

$$d_n^0 = \sum_{i=1}^n \dim \ker(-\Delta - \lambda_i^0 A_0), \quad d_n^\infty = \sum_{i=1}^n \dim \ker(-\Delta - \lambda_i^\infty A_\infty).$$

Our main results are the following theorems

**Theorem 3.3.** Suppose that  $F \in C^1(\Omega \times \mathbb{R}^2, \mathbb{R})$  satisfies (3.2) and  $(F_\infty^+)$ . Suppose moreover that there exists  $\beta \in \mathcal{M}_2(\Omega)$ ,  $\beta \preceq \lambda_{k+1}^\infty A_\infty$  such that

$$(\nabla F(x, \mathbf{z}_1) - \nabla F(x, \mathbf{z}_2)) \cdot (\mathbf{z}_1 - \mathbf{z}_2) \leq \beta(x)(\mathbf{z}_1 - \mathbf{z}_2) \cdot (\mathbf{z}_1 - \mathbf{z}_2), \quad (3.6)$$

then the system (3.1) has at least two nontrivial solutions in each of the following cases:

- (i)  $(F_0^+)$  holds with  $d_m^0 \neq d_k^\infty$ .
- (ii)  $(F_0^-)$  holds with  $d_{m-1}^0 \neq d_k^\infty$ .

**Theorem 3.4.** Suppose that  $F \in C^1(\Omega \times \mathbb{R}^2, \mathbb{R})$  satisfies (3.2) and  $(F_\infty^-)$ . Suppose moreover that there exists  $\beta \in \mathcal{M}_2(\Omega)$ ,  $\beta \succeq \lambda_{k-1}^\infty A_\infty$  such that

$$(\nabla F(x, \mathbf{z}_1) - \nabla F(x, \mathbf{z}_2)) \cdot (\mathbf{z}_1 - \mathbf{z}_2) \geq \beta(x)(\mathbf{z}_1 - \mathbf{z}_2) \cdot (\mathbf{z}_1 - \mathbf{z}_2), \quad (3.7)$$

then the system (3.1) has at least two nontrivial solutions in each of the following cases:



- (i)  $(F_0^+)$  holds with  $d_m^0 \neq d_{k-1}^\infty$ .
- (ii)  $(F_0^-)$  holds with  $d_{m-1}^0 \neq d_{k-1}^\infty$ .

Obviously, if in addition to (3.5) we have

$$|\nabla F(x, \mathbf{z}) - A_\infty(x)\mathbf{z}| = o(|\mathbf{z}|), \quad \text{as } |\mathbf{z}| \rightarrow \infty, \quad (3.8)$$

then (3.2) holds. In this case we say that the problem (3.1) is asymptotically linear at infinity. Since the pioneer work of Amann and Zehnder [2], asymptotically linear problems for a single equation have captured great interest. We refer to [16, 18] and references therein for some interesting results.

The asymptotically linear elliptic systems have captured some attentions in recent years. In [31, 32], the authors considered the case that  $A_0 = A_\infty$  are constant matrices. In [11], Furtado and Paiva studied the case that  $A_0$  and  $A_\infty$  are variable matrices. Under some conditions that ensure the Euler-Lagrange functional  $\Phi$  satisfying the Cerami type compactness condition, they obtained a nontrivial solution.

In [10], also for the case that  $A_0$  and  $A_\infty$  are variable matrices, Silva obtained two nontrivial solutions for the problem by applying Morse index type argument to  $\Phi$ . Thus it is essential to require  $F \in C^2$  so that  $\Phi$  is also of class  $C^2$ . Similar to [18], the reduction conditions (3.6) and (3.7) are used to control the Morse index. Finally, he also required some conditions to guarantee Cerami type compactness for  $\Phi$ .

Recently, Furtado and Paiva obtained a multiplicity result [12, Theorem 1.1] under (3.6) and  $(F_\infty^+)$ . But they only considered the case that  $1 \in (\lambda_m(A_0), \lambda_{m+1}(A_0))$  for some  $m \in \mathbb{N}$ , and they also required  $F \in C^2$ . Therefore, our Theorem 3.3 is an improvement.

Under our assumptions the functional  $\Phi$  may not satisfy the  $(PS)$  condition. To overcome this difficulty, as in [23, 25], we will perform the saddle point reduction and turn to consider the reduced functional  $\varphi$ . It turns out that the case of Theorem 3.4 is more difficult, because the reduced functional  $\varphi$  is defined on an infinite dimensional subspace.

Although our proof of Theorem 3.4 is based on some idea from [23], there are some real differences. The most significant one is that if  $A_0$  and  $A_\infty$  are different matrix functions, the local linking approach used in [23] does not work any more, see Remark 4.10 for details. Hence we must use critical groups and our abstract result (Theorem 1.1) is crucial.

## 4 Proofs of Theorems 3.3 and 3.4

Let  $\Phi$  be the functional introduced in (3.3). To prove our theorems it suffices to show that  $\Phi$  has two nonzero critical points.

Recall that for our Hilbert space  $X$  and  $p \in [2, 2^*]$ , by Sobolev inequality there is a constant  $S_p$  such that

$$|\mathbf{z}|_p := \left( \int_\Omega |\mathbf{z}|^p dx \right)^{1/p} \leq S_p \|\mathbf{z}\|. \quad (4.1)$$

That is, the embedding  $X \hookrightarrow L^p(\Omega) \times L^p(\Omega)$  is continuous. Moreover, using the Rellich-Kondrachov theorem we see that the embedding is also compact if  $p \in [2, 2^*)$ .



**Lemma 4.1.** (i) If  $(F_0^+)$  holds, then  $C_{d_m^0}(\Phi, \mathbf{0}) \neq 0$ .

(ii) If  $(F_0^-)$  holds, then  $C_{d_{m-1}^0}(\Phi, \mathbf{0}) \neq 0$ .

*Proof.* We only prove Case (ii). Set  $V_0 = \ker(-\Delta - \lambda_m^0 A_0)$ ,

$$V_- = \bigoplus_{i=1}^{m-1} \ker(-\Delta - \lambda_i^0 A_0), \quad V_+ = \overline{\bigoplus_{i=m+1}^{\infty} \ker(-\Delta - \lambda_i^0 A_0)}.$$

Then  $\dim V_- = d_{m-1}^0$ . We will show that  $\Phi$  has a local linking with respect to the decomposition  $X = V_- \oplus (V_0 \oplus V_+)$ . Namely, there exists  $\rho > 0$  such that

$$\begin{cases} \Phi(\mathbf{z}) \leq 0 & \text{for } \mathbf{z} \in V_-, \|\mathbf{z}\| \leq \rho, \\ \Phi(\mathbf{z}) > 0 & \text{for } \mathbf{z} \in V_0 \oplus V_+, 0 < \|\mathbf{z}\| \leq \rho. \end{cases} \quad (4.2)$$

The desired result will then follow from [19, Theorem 2.1]. To prove (4.2), we argue as in [17, Page 24].

Since  $\lambda_m^0 = 1$  is an isolated eigenvalue, it is well known that there exists positive number  $\kappa > 0$  such that

$$\pm \frac{1}{2} \int_{\Omega} (|\nabla \mathbf{z}|^2 - A_0(x) \mathbf{z} \cdot \mathbf{z}) \, dx \geq \kappa \|\mathbf{z}\|^2, \quad \mathbf{z} \in V_{\pm}.$$

Using (3.2), (3.5), there exists  $C > 0$  such that

$$|G(x, \mathbf{z})| \leq \frac{\kappa}{8S_2^2} |\mathbf{z}|^2 + C_1 |\mathbf{z}|^{2^*}, \quad (x, \mathbf{z}) \in \Omega \times \mathbb{R}^2. \quad (4.3)$$

For  $\mathbf{z} \in V_-$ , using (4.3) and (4.1), we obtain

$$\begin{aligned} \Phi(\mathbf{z}) &= \frac{1}{2} \int_{\Omega} (|\nabla \mathbf{z}|^2 - A_0(x) \mathbf{z} \cdot \mathbf{z}) \, dx - \int_{\Omega} G(x, \mathbf{z}) \, dx \\ &\leq -\kappa \|\mathbf{z}\|^2 + \frac{\kappa}{8S_2^2} |\mathbf{z}|_2^2 + C_1 |\mathbf{z}|_2^{2^*} \\ &\leq -\frac{\kappa}{2} \|\mathbf{z}\|^2 + C_3 \|\mathbf{z}\|^{2^*} \leq 0, \end{aligned} \quad (4.4)$$

provided  $\|\mathbf{z}\| \leq \rho_1 = (2^{-1} C_3^{-1} \kappa)^{1/(2^*-2)}$ .

On the other hand, since  $\dim V_0 < \infty$ , there exists  $C_2 > 0$  such that

$$\|\mathbf{v}\|_{\infty} \leq C_2 \|\mathbf{v}\|, \quad \text{for } \mathbf{v} \in V_0.$$

For  $\mathbf{z} \in V_0 \oplus V_+$  with  $\|\mathbf{z}\| \leq 2^{-1} C_2^{-1} \delta$ , we may write  $\mathbf{z} = \mathbf{v} + \mathbf{w}$ , where  $\mathbf{v} \in V_0$ ,  $\mathbf{w} \in V_+$ . Set

$$\Omega_1 = \left\{ x \in \Omega \mid |\mathbf{w}(x)| \leq \frac{\delta}{2} \right\}, \quad \Omega_2 = \Omega \setminus \Omega_1.$$

For  $x \in \Omega_2$ , we have

$$\begin{aligned} |\mathbf{z}(x)| &\leq |\mathbf{v}(x)| + |\mathbf{w}(x)| \leq \|\mathbf{v}\|_{\infty} + |\mathbf{w}(x)| \\ &\leq C_2 \|\mathbf{v}\| + |\mathbf{w}(x)| \end{aligned}$$

$$\leq C_2 \|\mathbf{z}\| + |\mathbf{w}(x)| \leq \frac{\delta}{2} + |\mathbf{w}(x)| \leq 2|\mathbf{w}(x)|.$$

By (4.3), we see that for  $x \in \Omega_2$ ,

$$G(x, \mathbf{z}) \leq \frac{\kappa}{8S_2^2} |\mathbf{z}|^2 + C_1 |\mathbf{z}|^{2^*} \leq \frac{\kappa}{2S_2^2} |\mathbf{w}|^2 + C'_1 |\mathbf{w}|^{2^*}.$$

This is also true for  $x \in \Omega_1$ , because in this case

$$|\mathbf{z}(x)| \leq |\mathbf{v}(x)| + |\mathbf{w}(x)| \leq |\mathbf{v}|_\infty + \frac{\delta}{2} \leq C_2 \|\mathbf{v}\| + \frac{\delta}{2} \leq C_2 \|\mathbf{z}\| + \frac{\delta}{2} \leq \delta,$$

hence  $G(x, \mathbf{z}) \leq 0$  by our assumption  $(F_0^-)$ . Therefore using (4.1) we deduce

$$\begin{aligned} \Phi(\mathbf{z}) &= \frac{1}{2} \int_{\Omega} (|\nabla \mathbf{w}|^2 - A_0(x) \mathbf{w} \cdot \mathbf{w}) \, dx - \int_{\Omega} G(x, \mathbf{z}) \, dx \\ &\geq \kappa \|\mathbf{w}\|^2 - \frac{\kappa}{2S_2^2} |\mathbf{w}|_2^2 - C'_1 |\mathbf{w}|^{2^*} \\ &\geq \frac{\kappa}{2} \|\mathbf{w}\|^2 - C_4 \|\mathbf{w}\|^{2^*}, \quad \mathbf{z} = \mathbf{v} + \mathbf{w} \in V_0 \oplus V_+. \end{aligned} \quad (4.5)$$

Now, let  $\mathbf{z} \in V_0 \oplus V_+$  be such that

$$0 < \|\mathbf{z}\| \leq \rho_2 = \min \left\{ \frac{\delta}{2C_2}, \left( \frac{\kappa}{2C_4} \right)^{1/(2^*-2)} \right\}.$$

If  $\mathbf{w} \neq 0$ , since  $\|\mathbf{w}\| \leq \|\mathbf{z}\|$ , by (4.5) we may deduce  $\Phi(\mathbf{z}) > 0$ . If  $\mathbf{w} = 0$ , then  $\mathbf{z} \in V_0$  and  $|\mathbf{z}|_\infty \leq C_2 \|\mathbf{z}\| \leq \delta$ , using  $(F_0^-)$  again, we also have

$$\Phi(\mathbf{z}) = - \int_{\Omega} G(x, \mathbf{z}) \, dx = - \int_{|\mathbf{z}| \leq \delta} G(x, \mathbf{z}) \, dx > 0.$$

Combining the above argument, we see that (4.2) is true with  $\rho = \min\{\rho_1, \rho_2\}$ .  $\square$

*Remark 4.2.* If we replace  $(F_0^-)$  by the weaker condition:  $G(x, \mathbf{z}) \leq 0$  for  $|\mathbf{z}| \leq \delta$ , then we can only obtain  $\Phi(\mathbf{z}) \geq 0$  in the second line of (4.2). Namely, we only have a weak local linking in the sense of Brezis and Nirenberg [5], which is not sufficient to obtain Lemma 4.1 via the result of J.Q. Liu [19]. However, if (3.2) is replaced by the stronger condition

$$|\nabla F(x, \mathbf{z}_1) - \nabla F(x, \mathbf{z}_2)| \leq \Lambda |\mathbf{z}_1 - \mathbf{z}_2|, \quad x \in \Omega, \quad \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^2,$$

then the functional  $\Phi$  is of class  $C^{2-0}$ . According to Perera [28, Theorem 2.6], we can still obtain the conclusion of Lemma 4.1.

As mentioned before, the proof of Theorem 3.4 is more difficult. Therefore, in what follows we will only prove Theorem 3.4. Let

$$X^- = \overline{\bigoplus_{i=k}^{\infty} \ker(-\Delta - \lambda_i^\infty A_\infty)}, \quad X^+ = \bigoplus_{i=1}^{k-1} \ker(-\Delta - \lambda_i^\infty A_\infty). \quad (4.6)$$

To verify the condition  $(A_-)$  and perform saddle point reduction, we need the following result.

**Proposition 4.3** ([10, Proposition 3.9 (b)]). *Let  $\beta \in \mathcal{M}_2(\Omega)$ ,  $\beta \succeq \lambda_{k-1}^\infty A_\infty$ . Then there exists  $\delta > 0$  such that*

$$-\|\mathbf{z}\|^2 + \int_{\Omega} \beta(x) \mathbf{z} \cdot \mathbf{z} \, dx \geq \delta \|\mathbf{z}\|^2, \quad \text{for all } \mathbf{z} \in X^+.$$

*Remark 4.4.* There is also a similar result for the dual case  $\beta \preceq \lambda_{k+1}^\infty A_\infty$ , see [10, Proposition 3.9 (a)]. This will be needed in the proof of Theorem 3.3.

Now, for  $\mathbf{v} \in X^-$  and  $\mathbf{w}_1, \mathbf{w}_2 \in X^+$ , using Proposition 4.3 and our assumption (3.7) we obtain

$$\begin{aligned} & -\langle \nabla \Phi(\mathbf{v} + \mathbf{w}_1) - \nabla \Phi(\mathbf{v} + \mathbf{w}_2), \mathbf{w}_1 - \mathbf{w}_2 \rangle \\ &= -\int_{\Omega} |\nabla(\mathbf{w}_1 - \mathbf{w}_2)|^2 \, dx + \int_{\Omega} (\nabla F(x, \mathbf{v} + \mathbf{w}_1) - \nabla F(x, \mathbf{v} + \mathbf{w}_2)) \cdot (\mathbf{w}_1 - \mathbf{w}_2) \, dx \\ &\geq -\int_{\Omega} |\nabla(\mathbf{w}_1 - \mathbf{w}_2)|^2 \, dx + \int_{\Omega} \beta(x) (\mathbf{w}_1 - \mathbf{w}_2) \cdot (\mathbf{w}_1 - \mathbf{w}_2) \, dx \\ &\geq \delta \|\mathbf{w}_1 - \mathbf{w}_2\|^2. \end{aligned}$$

Therefore,  $\Phi$  satisfies the condition  $(A_-)$  and we obtain a reduced functional  $\varphi : X^- \rightarrow \mathbb{R}$ , which is of class  $C^1$ . It suffices to find two non-zero critical points of  $\varphi$ .

We want to show that  $\varphi$  is coercive. For this, it is quite natural to pick a sequence  $\{\mathbf{v}_n\}$  in  $X^-$  such that  $\|\mathbf{v}_n\| \rightarrow \infty$ . Then consider the normalization sequence  $\{\|\mathbf{v}_n\|^{-1} \mathbf{v}_n\}$ . However, since  $\dim X^- = \infty$ , the weak limit of the normalization sequence may be the zero element in  $X^-$ . This makes it difficult to prove that  $\varphi(\mathbf{v}_n) \rightarrow +\infty$ .

To get around this difficulty, as in [23] we consider  $\Phi_1$ , the restriction of  $\Phi$  on  $X^-$ . Then  $\Phi_1 \in C^1(X^-, \mathbb{R})$ . The following ‘non vanishing lemma’ is the key ingredient of our approach.

**Lemma 4.5.** *Let  $\{\mathbf{v}_n\}$  be a sequence in  $X^-$  such that  $\Phi_1(\mathbf{v}_n) \leq c$  and  $\|\mathbf{v}_n\| \rightarrow \infty$ . Denote  $\mathbf{v}_n^0 = \|\mathbf{v}_n\|^{-1} \mathbf{v}_n$ . Then there is a subsequence of  $\{\mathbf{v}_n^0\}$  which converges weakly to some point  $\mathbf{v}^0 \neq \mathbf{0}$ .*

*Proof.* The proof is quite similar to that of [23, Lemma 3.2], where instead of  $\Phi_1(\mathbf{v}_n) \leq c$ , it is assumed that  $\nabla \Phi_1(\mathbf{v}_n) \rightarrow 0$ . Since  $\{\mathbf{v}_n^0\}$  is bounded, up to a subsequence, we may assume that  $\mathbf{v}_n^0 \rightharpoonup \mathbf{v}^0$  in  $X^-$ . The compactness of the embedding

$$X^- \hookrightarrow X \hookrightarrow L^2(\Omega) \times L^2(\Omega)$$

implies that  $\mathbf{v}_n^0 \rightarrow \mathbf{v}^0$  in  $L^2(\Omega) \times L^2(\Omega)$ . By (3.2) we have

$$|F(x, \mathbf{z})| \leq \frac{1}{2} \Lambda |\mathbf{z}|^2, \quad (x, \mathbf{z}) \in \Omega \times \mathbb{R}^2.$$

Therefore

$$\begin{aligned} 2c &\geq 2\Phi_1(\mathbf{v}_n) = \int_{\Omega} |\nabla \mathbf{v}_n|^2 \, dx - \int_{\Omega} 2F(x, \mathbf{v}_n) \, dx \\ &\geq \int_{\Omega} |\nabla \mathbf{v}_n|^2 \, dx - \Lambda \int_{\Omega} |\mathbf{v}_n|^2 \, dx \\ &= \|\mathbf{v}_n\|^2 - \Lambda \|\mathbf{v}_n\|_2^2. \end{aligned}$$

Multiplying by  $\|\mathbf{v}_n\|^{-2}$  on both sides, we deduce

$$2c \|\mathbf{v}_n\|^{-2} \geq 1 - \Lambda |\mathbf{v}_n^0|_2^2.$$

Since  $|\mathbf{v}_n^0|_2 \rightarrow |\mathbf{v}^0|_2$  and  $\|\mathbf{v}_n\|^{-2} \rightarrow 0$ , the above inequality implies that  $|\mathbf{v}^0|_2^2 \geq \Lambda^{-1}$  and hence  $\mathbf{v}^0 \neq \mathbf{0}$ .  $\square$

*Remark 4.6.* This is the only place where we need (3.2). In the case of Theorem 3.3, the reduced functional  $\varphi$  is defined on a finite dimensional subspace. Hence it is easy to obtain the coerciveness of  $\varphi$  using the assumption  $(F_\infty^+)$ . Therefore, in Theorem 3.3, we may replace (3.2) with a subcritical growth condition.

**Lemma 4.7.** *The functional  $\Phi_1 : X^- \rightarrow \mathbb{R}$  is coercive, and bounded from below.*

*Proof.* Assume for a contradiction that for some  $\{\mathbf{v}_n\} \subset X^-$  and  $c > 0$  we have

$$\Phi_1(\mathbf{v}_n) \leq c, \quad \|\mathbf{v}_n\| \rightarrow \infty. \quad (4.7)$$

Let  $\mathbf{v}_n^0 = \|\mathbf{v}_n\|^{-1} \mathbf{v}_n$ , by Lemma 4.5, up to a subsequence we have  $\mathbf{v}_n^0 \rightharpoonup \mathbf{v}^0$  for some  $\mathbf{v}^0 \neq \mathbf{0}$ . Let

$$\Theta = \left\{ x \in \Omega \mid \mathbf{v}^0(x) \neq \mathbf{0} \right\},$$

then  $|\Theta| > 0$ . For  $x \in \Theta$  we have

$$|\mathbf{v}_n(x)| = \|\mathbf{v}_n\| \left| \mathbf{v}_n^0(x) \right| \rightarrow \infty.$$

By  $(F_\infty^-)$  and the Fatou Lemma,

$$\int_{\Theta} \left( \frac{1}{2} A_\infty(x) \mathbf{v}_n \cdot \mathbf{v}_n - F(x, \mathbf{v}_n) \right) dx \rightarrow +\infty, \quad \text{as } n \rightarrow \infty.$$

On the other hand,  $(F_\infty^-)$  also implies the existence of  $M > 0$  such that

$$\frac{1}{2} A_\infty(x) \mathbf{z} \cdot \mathbf{z} - F(x, \mathbf{z}) \geq -M, \quad (x, \mathbf{z}) \in \Omega \times \mathbb{R}^2. \quad (4.8)$$

Therefore,

$$\begin{aligned} \Phi_1(\mathbf{v}_n) &= \frac{1}{2} \int_{\Omega} |\nabla \mathbf{v}_n|^2 dx - \int_{\Omega} F(x, \mathbf{v}_n) dx \\ &\geq \int_{\Omega} \left( \frac{1}{2} A_\infty(x) \mathbf{v}_n \cdot \mathbf{v}_n - F(x, \mathbf{v}_n) \right) dx \\ &= \left( \int_{\Theta} + \int_{\Omega \setminus \Theta} \right) \left( \frac{1}{2} A_\infty(x) \mathbf{v}_n \cdot \mathbf{v}_n - F(x, \mathbf{v}_n) \right) dx \\ &\geq \int_{\Theta} \left( \frac{1}{2} A_\infty(x) \mathbf{v}_n \cdot \mathbf{v}_n - F(x, \mathbf{v}_n) \right) dx - M |\Omega \setminus \Theta| \rightarrow +\infty. \end{aligned}$$

This contradicts with (4.7). Thus  $\Phi_1$  is coercive. It follows that  $\Phi_1$  is bounded from below.  $\square$

*Remark 4.8.* In [23, Lemma 3.2], another version of ‘non vanishing lemma’ (as mentioned in the proof of Lemma 4.5) is proved and used in [23, Lemma 3.4] to show that  $\Phi_1$  satisfies the (PS) condition. Then the coerciveness of  $\Phi_1$  is obtained via a well-known result of Li [15], see also [6]. Our argument in Lemmas 4.5 and 4.7 does not involve the derivative information of  $\Phi_1$ , hence is considerably simpler.

**Lemma 4.9.** *Under the assumption of Theorem 3.4, the functional  $\varphi$  is bounded from below. Moreover,  $\varphi$  satisfies the (PS) condition.*

*Proof.* Let  $K : X \rightarrow X$  be defined as

$$\langle Kz, w \rangle = \int_{\Omega} \nabla F(x, z) \cdot w \, dx.$$

Then  $K$  is compact and  $\nabla \Phi = \mathbf{1}_X - K$ . Obviously  $\nabla \Phi$  maps bounded sets to bounded sets. By Proposition 2.1,  $\nabla \varphi$  is also a compact perturbation of  $\mathbf{1}_{(X^-)}$ .

By the definition of the reduced functional  $\varphi$ , we have

$$\varphi(v) = \max_{w \in X^+} \Phi(v + w) \geq \Phi(v) = \Phi_1(v).$$

Using Lemma 4.7 we see that  $\varphi$  is also coercive and bounded from below. In particular, any (PS) sequence of  $\varphi$  is bounded. Applying [29, Proposition 2.2], we deduce that  $\varphi$  satisfies (PS).  $\square$

*Proof of Theorem 3.4.* We prove the case (i). By Lemma 4.9,  $\varphi$  satisfies the (PS) condition, and bounded from below. Note that

$$\mu = \dim X^+ = d_{k-1}^{\infty},$$

by Theorem 1.1 and Lemma 4.1 we obtain

$$C_{d_m^0 - d_{k-1}^{\infty}}(\varphi, \mathbf{0}) \cong C_{d_m^0}(\Phi, \mathbf{0}) \neq 0.$$

Now, if  $d_m^0 \neq d_{k-1}^{\infty}$ , the desired result follows from Proposition 2.2.  $\square$

*Remark 4.10.* If  $A_0 = A_{\infty}$ , then the decompositions of  $X$  in Lemma 4.1 and in (4.6) are related to the same eigenvalue problem (3.4). For the case proved in this section, namely Theorem 3.4 (ii), we have  $m > k$  and  $V_0 \oplus V_+ \subset X^-$ .

As in [23], it is then easy to show that  $\varphi$  has a local linking with respect to the decomposition  $X^- = (V_- \cap X^-) \oplus (V_0 \oplus V_+)$ . Then the local linking version of the three critical points theorem [5, 20] yields the desired result, we don’t need Theorem 1.1.

On the other hand, if  $A_0 \neq A_{\infty}$ , then the above inclusion of the decompositions is false, the local linking property of  $\Phi$  does not descend to  $\varphi$ . Hence the local linking version of the three critical points theorem is not applicable, and our Theorem 1.1 is crucial.

**acknowledgements** This work was completed while S.B. Liu was visiting the Institute of Mathematics, Chinese Academy of Sciences; and the Institute of Mathematics, Peking University. S.B. Liu would like to thank both institutes for invitation and hospitality.

## References

- [1] Amann, H.: [Saddle points and multiple solutions of differential equations](#). Math. Z. **169**, 127–166 (1979)
- [2] Amann, H., Zehnder, E.: [Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations](#). Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7**, 539–603 (1980)
- [3] Arcoya, D., Costa, D.G.: Nontrivial solutions for a strongly resonant problem. Differential Integral Equations **8**, 151–159 (1995)
- [4] Bartsch, T., Li, S.: [Critical point theory for asymptotically quadratic functionals and applications to problems with resonance](#). Nonlinear Anal. **28**, 419–441 (1997)
- [5] Brezis, H., Nirenberg, L.: [Remarks on finding critical points](#). Comm. Pure Appl. Math. **44**, 939–963 (1991)
- [6] Čaklović, L., Li, S.J., Willem, M.: A note on Palais-Smale condition and coercivity. Differential Integral Equations **3**, 799–800 (1990)
- [7] Castro, A., Cossio, J.: [Multiple solutions for a nonlinear Dirichlet problem](#). SIAM J. Math. Anal. **25**, 1554–1561 (1994)
- [8] Castro B., A.: Reduction methods via minimax. In Differential equations (S ao Paulo, 1981), *Lecture Notes in Math.*, volume 957, 1–20, Springer, Berlin, 1982
- [9] Chang, K.c.: Infinite-dimensional Morse theory and multiple solution problems. Progress in Nonlinear Differential Equations and their Applications, 6, Birkhäuser Boston Inc., Boston, MA, 1993
- [10] Da Silva, E.D.: Multiplicity of solutions for gradient systems. Electron. J. Differential Equations No. 64, 15 (2010)
- [11] Furtado, M.F., de Paiva, F.O.V.: [Multiplicity of solutions for resonant elliptic systems](#). J. Math. Anal. Appl. **319**, 435–449 (2006)
- [12] Furtado, M.F., de Paiva, F.O.V.: [Multiple solutions for resonant elliptic systems via reduction method](#). Bull. Aust. Math. Soc. **82**, 211–220 (2010)
- [13] Lazer, A.C., McKenna, P.J.: [Multiplicity results for a class of semilinear elliptic and parabolic boundary value problems](#). J. Math. Anal. Appl. **107**, 371–395 (1985)
- [14] Li, C., Li, S., Liu, J.: [Splitting theorem, Poincaré-Hopf theorem and jumping nonlinear problems](#). J. Funct. Anal. **221**, 439–455 (2005)
- [15] Li, S.: Some existence theorems of critical points and applications. Technical Report IC/86/90, ICTP, Trieste (1986)
- [16] Li, S., Liu, J.Q.: Computations of critical groups at degenerate critical point and applications to nonlinear differential equations with resonance. Houston J. Math. **25**, 563–582 (1999)

- [17] Li, S., Willem, M.: [Applications of local linking to critical point theory](#). J. Math. Anal. Appl. **189**, 6–32 (1995)
- [18] Li, S., Willem, M.: [Multiple solutions for asymptotically linear boundary value problems in which the nonlinearity crosses at least one eigenvalue](#). NoDEA Nonlinear Differential Equations Appl. **5**, 479–490 (1998)
- [19] Liu, J.Q.: The Morse index of a saddle point. Systems Sci. Math. Sci. **2**, 32–39 (1989)
- [20] Liu, J.Q., Li, S.J.: An existence theorem for multiple critical points and its application. Kexue Tongbao (Chinese) **29**, 1025–1027 (1984)
- [21] Liu, J., Su, J.: [Remarks on multiple nontrivial solutions for quasi-linear resonant problems](#). J. Math. Anal. Appl. **258**, 209–222 (2001)
- [22] Liu, S.: [Remarks on multiple solutions for elliptic resonant problems](#). J. Math. Anal. Appl. **336**, 498–505 (2007)
- [23] Liu, S.: [Multiple solutions for elliptic resonant problems](#). Proc. Roy. Soc. Edinburgh Sect. A **138**, 1281–1289 (2008)
- [24] Liu, S., Li, S.: [Critical groups at infinity, saddle point reduction and elliptic resonant problems](#). Commun. Contemp. Math. **5**, 761–773 (2003)
- [25] Liu, S.Q., Tang, C.L., Wu, X.P.: [Multiplicity of nontrivial solutions of semilinear elliptic equations](#). J. Math. Anal. Appl. **249**, 289–299 (2000)
- [26] Long, Y.M.: Maslov-type index, degenerate critical points, and asymptotically linear Hamiltonian systems. Sci. China Ser. A **33**, 1409–1419 (1990)
- [27] Mawhin, J., Willem, M.: Critical point theory and Hamiltonian systems, *Applied Mathematical Sciences*, volume 74. Springer-Verlag, New York, 1989
- [28] Perera, K.: [Critical groups of critical points produced by local linking with applications](#). Abstr. Appl. Anal. **3**, 437–446 (1998)
- [29] Struwe, M.: Variational methods, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, volume 34. Springer-Verlag, Berlin, 1996, 2nd edition, applications to nonlinear partial differential equations and Hamiltonian systems
- [30] Wang, Z.Q.: A note on the second variation theorem. Acta Math. Sinica **30**, 106–110 (1987)
- [31] Zou, W.: [Multiple solutions for asymptotically linear elliptic systems](#). J. Math. Anal. Appl. **255**, 213–229 (2001)
- [32] Zou, W., Li, S., Liu, J.Q.: [Nontrivial solutions for resonant cooperative elliptic systems via computations of critical groups](#). Nonlinear Anal. **38**, 229–247 (1999)